A NOTE ON THE EVOLUTION OF SYMBOLS IN ALGEBRA

My intention of writing this note is to instill some insight into the difficulties that the great mathematicians of the past faced due to lack of proper tools. In fact, a topic such as 'The History of Mathematical Symbols' would have been ideal, but it is too enormous to deal with, and many good books are available on this topic. Here, I intend to touch upon only a few symbols in Algebra, which have some interesting histories. By Algebra, I mean the elementary algebra, which includes Solution of Equations, Number Theory and Trigonometry, but excludes modern topics such as Group Theory and recent axiomatic Algebras. Most of these new branches came up during the 20th century and not enough time has passed for these to get into History.

According to [5], the origin of the word algebra is from the title of Al Khowarizmi's treatise *Hisab al-jabr w'al-muqa-balah* written in Arabic. This title when translated into English means "The science of the transposition and the cancellation". The Arabic term al-jabr, used as a name for 'the science of equation' entered Europe through the moors of Spain. In Spain an *Algebrista* is a bone-setter and sometimes this term was used for barbers as they did bone-setting as a sideline. A 15th century manuscript referred to this science as 'Gebra and Almuthabola'. Some Arab writers thought that there was an Indian mathematician by the name Arjabahr.See [6] pages 386 - 392 for various other names used earlier for 'Algebra'.

First of all let us address ourselves to a few preliminaries such as:

- Why should we bother to study history?
- Who are the recent reputed historians?
- What are the sources for the conclusions they have arrived at?
- From where do these historians gather information?
- What information is available, about people, terms and symbols?
- From where can we gather such information?

Generally, historians study two types of history; one is termed the cyclic history and the other is linear history. Cyclic history is useful in politics, in warfare, in forecasting etc. This is where we say history repeats itself. History of Mathematics is linear, like in the search of ancestry. In our study of history we are interested only in finding who proved what, and why he proved it, rather than how he proved it. This will lead us to understand the original difficulties the great Mathematicians of yesteryears had in their attempts to solve mathematical mysteries. Also, this will guide a researcher in the direction of a specific goal rather than make him to wander in obscurity. However, as teachers, our intention of learning the origins of terms and symbols is different. The students are always curious. If a student asks, why the letter *m* is used for gradient or why 360 is taken for subdividing the full circle into degrees, what is the answer you would give? "I do not know " is definitely not the answer. Therefore, one of the reasons why we should know the history is to enlighten the students and supplement the subject matter whenever possible with historical interludes.

Florian Cajori is an authority on history of Mathematics, and he has written two volumes on mathematical notations, in addition to two other books on history of Mathematics. C.B. Boyer, Rouse Ball, E.T. Bell, Howard Eves and D.E. Smith are a few Mathematics historians whose works are available in the local university libraries. 'An Introduction to the History of Mathematics' by Howard Eves is a magnificent textbook meant for a degree course in that subject and I have used this book extensively in preparing these notes.

It is worth referring to the books by Boyer, Bell, Smith and Eves for gathering general knowledge in history. There are other recent books on this subject such as 'History of Mathematical Symbols' by Johnson, Dale Seymour Publications, 1994. The Internet is another good source of information. This is a better source as the information given there is updated regularly. The information I use here was updated in May 2003. The sites I have used most are http://www.st-and.ac.uk/history/Bibliography/index.html, http://www.Earth/matriX.com and the web-site of the University of Virginia, USA.

Mathematics of the Babylonians is found written in a collection of papyri found at different sites and different times. Some famous ones are:

- Haris papyrus 1167 BC prepared by Ramses V.
- Kahun papyrus 1950 BC found at Kahun.
- Moscow papyrus 1850 BC purchased in Egypt in 1983.
- Rhind papyrus 1650BC purchased by Scottish Egyptologist Rhind in 1858.
- (You may refer to [5] to find how these papyri got their names.)

Since the first half of the 19th century, archeologists working in Mesopotamia have unearthed some half-million inscribed clay tablets of which about 300 are on Mathematics. Some of these tablets are in the great museums in Paris, Berlin and London and universities such as Yale, Colombia, Pennsylvania. Most of the Mathematics history of Egypt is due to Otto Neugebauer and Thureau-Dangin who studied these tablets. Most of the knowledge that was gathered from the contents of these tablets was not known even in the latter half of the 19th century. Interpretations of these tablets are still going on. Some of the tablets are named (depending on where these are located) as Louvre Tablets, Plimton 322 Tablets, Susa Tablet, Yale tablets etc.

The other sources were the stone pillars and obelisks found at various places. For example the Ashoka pillars in India established by King Ashoka (272- 232 BC) indicated the use of the present day numerals. These symbols were copied by the Arabs and are now known as the Hindu-Arab numerals.

As a preliminary to the discussion, it should be noted that in 1842, G.H.F. Nesselmann divided the history of algebra into three periods - the rhetorical, in which the problem as well as the solution are expressed in words in full and sometimes in verse form. the next is known as the syncopated, where the words are replaced by abbreviations and finally, the symbolic, in which the abbreviations and operations were replaced by symbols. This is the Algebra we are familiar with. The solution of a problem given in rhetoric requires making a clear mental picture of the whole situation described in the problem, concoct several chains of reasoning, and arrive at the conclusion by linking these chains logically. If there are several chains the solver might go insane when he tries to remember them all.

A well known example of a rhetorical problem is the famous cattle problem of Archemedes which reads as:

If thou art diligent and wise, O stranger, compute the number of cattle of the Sun, who once upon a time grazed on the fields of the Thrinacian isle of Sicily, divided into four herds of different colours, one milk white, another a glossy black, a third yellow and the last dappled. In each herd were bulls, mighty in number according to these proportions: Understand, stranger, that the white bulls were equal to a half and a third of the black together with the whole of the yellow, while the black were equal to the fourth part of the dappled and a fifth, together with, once more, the whole of the yellow. Observe further that the remaining bulls, the dappled, were equal to a sixth part of the white and a seventh, together with all of the yellow. These were the proportions of the cows: The white were precisely equal to the third part and a fourth of the whole herd of the black; while the black were equal to the fourth part once more of the dappled and with it a fifth part, when all, including the bulls, went to pasture together. Now the dappled in four parts were equal in number to a fifth part and a sixth of the yellow herd. Finally the yellow were in number equal to a sixth part and a seventh of the white herd. If thou canst accurately tell, O stranger, the number of cattle of the Sun, giving separately the number of well-fed bulls and again the number of females according to each colour, thou wouldst not be called unskilled or ignorant of numbers, but not yet shalt thou be numbered among the wise.

But come, understand also all these conditions regarding the cattle of the Sun. When the white bulls mingled their number with the black, they stood firm, equal in depth and breadth, and the plains of Thrinacia, stretching far in all ways, were filled with their multitude. Again, when the yellow and the dappled bulls were gathered into one herd they stood in such a manner that their number, beginning from one, grew slowly greater till it completed a triangular figure, there being no bulls of other colours in their midst nor none of them lacking. If thou art able, O stranger, to find out all these things and gather them together in your mind, giving all the relations, thou shalt depart crowned with glory and knowing that thou hast been adjudged perfect in this species of wisdom.

Obviously, the solution in rhetoric is extremely tedious (or impossible) and surely, Archemedes may never have expected anyone to try it. The complete answer to the first part of this problem solved using symbols is found at

http://www.mcs.drexel.edu/~crorres/Archimedes/Cattle/Solution1.html

The second part of the problem reduces to the problem of finding positive integers r and m which satisfy the equation $102{,}571{,}605{,}819{,}606\ r^2 = m(m + 1)/2$

Amthor found a partial solution to this problem. He found that there are infinite number of solutions and the smallest one is an integer, which contains 206,545 digits, and begins with the digits 776. "Since it has been calculated that it would take the work of a thousand men for a thousand years to determine the complete number of cattle, it is obvious that the world will never have a complete solution."

An ad hoc group called the Hillsboro Mathematical Club (Illinois, USA) in the years 1889 to 1893 continued Amthor's calculations and computed the first 31 digits and the last 12 digits of the smallest total number of cattle and found them to be

7760271406486818269530232833209 . . . 719455081800.

For more details regarding this solution, and the final result obtained by using a Cray computer can be seen at

http://www.mcs.drexel.edu/~crorres/Archimedes/Cattle/Solution2.html

The first writer to make use of algebraic notation to express algebraic expressions was Diophantus. During his time the numbers were written using the Greek alphabet as follows:

$\alpha = 1, \beta = 2, \gamma = 3, \delta = 4, \varepsilon = 5, \vartheta = 6, \zeta = 7, \eta = 8, \theta = 9,$

$\iota = 10, \kappa = 20, \lambda = 30, \mu = 40, \nu = 50, \ldots\ldots$

$\rho = 100, \sigma = 200, \tau = 300, \ldots..$

For example $\iota\gamma$ and $\rho\lambda\zeta$ stand for 13 and 137 respectively.

(For the full set see [5] page 9. This is called the Ionic numeral system)

The following symbols were used for various powers of the variable. The variable itself is denoted by S. In fact, this is a symbol that looks like S, which was formed by merging a and r the first two letters of *arithmos* (number). The square of the variable is written as Δ^{γ} by merging the first two Greek letters of the word *Dunamis* (Power), and the cube of the variable is written as K^{γ} by using the first two Greek letters of the words *Kubos*. (Note that the second letter of each word is written slightly raised). The symbols $\Delta^{\gamma}\Delta$, DK°, $\Delta K^{\gamma}K$ were used to mean x^4, x^5 and x^6 respectively.

The constant term is indicated by M° which is formed by the first two Greek letters of the word Monados (unit). The symbol $\uparrow$ (*leipis*) is used for 'minus'. All the negative terms in an expression are grouped together at the end and preceded by $\uparrow$. For example $K^{\gamma}\alpha\Delta^{\gamma}\tau\gamma S\varepsilon$ and $\Delta K^{\gamma}\beta S\eta\uparrow\varepsilon M^{\circ}\alpha$ stand for the expressions $x^3 + 13\ x^2 + 5x$ and $2x^5 - 5\ x^2 + 8x - 1$ respectively.

However, in speaking of Diophantus, it should be remembered that the presently available manuscript was written about a thousand years after Diophantus wrote the original manuscript.

Similarly, the ancient Sinhalese used the letters of the Sinhala alphabet to represent the numbers. This system was called the 'Katapayadhee' system. Apart from these, there were several ways; that are specific to the era or the person, of syncopating mathematical expressions. Some of these are given in [6] pages 427 - 431.

During the 16th century, the algebra was syncopated by the use of p (piu – more) for plus, *m* (meno – less) for minus, co (cosa –thing) for the unknown x, ce(censo) for x^2, cu (cubus) for x^3 etc. the word *aequalis* which stands for equal was syncopated by Vieta to aeq. Cardan solved the cubic equation using this syncopation. Thus, the equation $x^3 + 6x = 20$ was written as cu *p* 6 co aeq 20. At last Descarte (1637) settled the matter by writing x, x^2, x^3, etc. for the powers of the variable x.

According to De Morgan, Algebraic symbols enabled those who are not great mathematicians in their generation to do, without effort, Mathematics, which would have baffled the great Mathematicians who were their predecessors. According to Bell, the formulae in an engineering handbook, if rewritten with abbreviations and convenient signs (*i.e.* syncopated version) might be intelligent to an Archemedian but to an average engineer they would probably be gibberish.

Let us solve two simple problems - one in arithmetic and the other in logic that are given in the rhetorical form.

1. How can you bring up from the river exactly six quarts of water when you have only two containers of capacity 9 quarts and 4 quarts to measure with.

Solution (rhetorical):

First fill the 9 quarts container. Using the 4 quarts container remove 8 quarts of water from it. The bigger container now contains 1 quart of water. Put this into the smaller one. Fill the 9 quarts container again and with this water completely fill the 4 quarts container, which contains 1 quart of water. Now the bigger container will be left with 6 quarts of water.

Solution(symbolic):

Let x be the number of times we have to fill the 9 quarts container and y be the number of times we have to fill the 4 quarts container to get the required amount of water. The problem now reduces to the following. Find integers x and y such that $9x + 4y = 6$.

Clearly $x = 2$, $y = -3$ is a solution. Here, 2 of 9 quarts means: Fill the 9 quarts container twice and -3 means: Put back to the river 3 times the capacity of the 4 quarts container.

We see that the rhetorical solution and the symbolic solution give the same answer. However, using the symbolic solution we find that there are infinite number of solutions given by $x = 2 + 4t$, $y = -3 - 9t$, where t is any integer. For example, $x = -2$ and $= 6$ is a solution. As an exercise describe this solution verbally as above (rhetorical form).

2. Ananda, Bandula and Channa eat at the office canteen. Each eats either rice or noodles.

(i) If Ananda eats rice, then Bandula eats noodles.

(ii) Either Ananda or Channa eats rice.

(iii) Bandula and Channa do not eat noodles at the same time.

Who could have eaten rice yesterday and noodles today?

Solution (rhetorical):

Suppose Ananda eats rice then by *(i)* Bandula eats noodles and by *(iii)* Channa has to eat rice. But this contradicts *(ii)*. Therefore, Ananda cannot eat rice and he has to eat noodles all the time. Next consider Channa. If he eats noodles then by *(ii)* Ananda has to eat rice. This is not possible as shown above. Thus Channa eats rice only. Thus only Bandula can have a choice.

Solution (symbolic):

$\sim(\sim b \wedge \sim c)$, Using the symbols of Symbolic Logic, we can write the three statements as $a \rightarrow \sim b$, $a \veebar c$ and $(\sim b \wedge \sim c)$, where a, b, c stand for the statements Ananda eats rice, Bandula eats rice, and Channa eats rice respectively. From these we deduce that $\sim a \wedge c$. Thus only Bandula can eat rice or noodles.

Broadly, the symbols in Algebra can be subdivided into symbols in Number Theory, symbols in Set Theory and Logic, symbols in Calculus and symbols in Trigonometry.

In general the symbols in each of these subtopics can be divided into six classes as

(i) Symbols of operation such as +, -, x, ÷, √ .

(ii) Grouping symbols such as ( ),{ } ______,

(iii) Symbols of relations such as =, <, >

(iv) Symbols for variables such as x, y, z.

(v) Symbols for various constants $\pi, \gamma, e, i, \ldots$

(vi) Symbols used to abbreviate such as f(x), sin, cos, lim, grad, div

For example let us consider the equation $ax^2 + bx + c = 0$, and discuss the words and symbols that arise from such an equation. The words are 'coefficients', 'variable' or 'unknown', 'power', 'degree', 'product', 'constant', and the symbols are + and =.

The solutions of the above equation are $x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$, and the new symbols

involved are -, ±, √ and the vinculum (*i.e.* the horizontal bar which separates the numerator from the denominator. Originally the word 'equation' was used (*c* 300*BC*) by Euclid to denote a continued proportion. The concept of Equations was not known to Babylonians.

In the present day context this word was used first by Gosslin in 1577. The symbols + and – were first used in the present day context by Widmann, a lecturer at the University of Leipzig in 1486. These symbols came into use in England after William Oughtred used them in 1627. Oughtred wrote a book named 'Clavis Mathematics', and this book contained over 150 new mathematical symbols. Out of these only 3 came down to present time. They are x, :: and ~

(for difference). The symbols + and – were in use before, for example to indicate whether a barrel is full or empty. The use of x for multiplication was due to Thomas Harriet (*c*1600). However, it became popular only after Leibniz used it in 1698. In a manuscript written in the 8th or 9th century, found buried in the village called Bakshali, in India, it is found that multiplication was indicated by placing integers side-by-side. According to Lucas, Michael Stifel (1544) was the first to show multiplication by juxtaposition, but according to Bell, Descarte (1637) was the first. In 1663 Harriett wrote a^2 as aa, a^3 as aaa etc. Pierre Harrigone used the symbols a^2, a^3 etc. in the 5 volumes he wrote as 'Cursus Mathematica'. Negative integers as exponents were first used in 1676 and fractions as exponents were first used in 1656, by Isaac Newton. The equal sign ' = ' was introduced by Robert Recorde in 1557. He justified the use of this symbol by saying " because no two things can be more equal".

In the study of the general equation, it was Napier who recognized the advantage of always equating to zero. The Encyclopaedia Britannica says, "Hindu literature gives evidence that the zero has been used before the birth of Christ. According to Johnson, by the 3rd century AD Hindu Mathematicians were using a heavy dot to mark its place in calculations and an empty circle eventually replaced the dot. According to Bell, it was used in India as seen in Pancasiddhantika by Varahamihira (505).

Viete coined the word 'coefficient' in the 16th century, the word 'constant' was coined by Leibniz, in the 17th century. The unknown quantity or the variable was called the hau by Ahmes(c1550 *BC*). Indians called it *yavat-tavat*. Arabs called it *sha* (meaning anything). The word *res* (thing) was also used in the medieval time. According to Burton, Brahmagupta and Bhaskara (1154) in writing fractions placed the numerator above the denominator, without a bar. (This bar is called the vinculum). When Arabs first copied the Hindu notation, they introduced the vinculum. The horizontal fraction bar is attributed to Al Hassar (1200). The Obelus ÷ was first used by Rahn in 1659. The radical sign $\sqrt{}$ first appeared in print in Rudolff's Coss (1525). He used this symbol, as it resembled r (first letter of radix). When Stifel edited this work in 1553, he introduced the symbols for cube root etc. Rene Descartė added the vinculum to the radical sign to group the symbols and numbers that should come under the radical sign.

The solution of the cubic equation $ax^3 + bx^2 + cx + d = 0$, was essentially completed by Cardan (1545). He published the complete solution in his book 'Ars Magna'. Since symbolic algebra was unknown at that time, the solution was very tedious to understand, as there are several possible cases to consider. It took 500 years to solve the cubic equation after the solution of the quadratic equation. The main reason for such a long period was the non-availability of proper tools of manipulation at that time. Now, the solution of the general cubic equation symbolically can be found in any book on advanced Algebra, (e.g. Algebra, Archibald) and can be comprehended by a first year undergraduate.

The perfection of algebraic symbolism was a major contributor to the speedy development of mathematics in the 17th century. On the other hand, the lack of an effective symbolism hampered the progress of mathematics during the periods of some great mathematicians such as Bhaskara and Al-Khowarizmi.

Finally, let us answer the question we asked at the beginning - why the letter *m* is used for gradient?

The answer given in the web page of St Andrew's University is as follows:

It is not known why the letter *m* was chosen for slope; the choice may have been arbitrary. John Conway has suggested *m* could stand for "modulus of slope." One high school algebra textbook says the reason for *m* is unknown, but remarks that it is interesting that the French word for "to climb" is *monter*. Several experts believe that there is no connection here. Descartes, who was French, did not use *m*.

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